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Shock propagation in variable area ducts with phase changes: an extension of Chisnell's method*

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SUMMARY

Chisnell's approximating ordinary differential equation for the motion of a shock in a variable area duct is generalized to allow for phase changes behind the shock.

1. Introduction

For systems of conservation laws of the form $u_t + f_x(u) = 0$, the propagation of shock and rarefaction waves is rather well understood. A general theory is given in [5], while specific problems are studied in detail in [2] and [10]. In [7] a theory of shock and rarefaction waves for anomalous gases is presented.

In all of these works it is crucial that the conservation law has exactly the form given above; namely, there are no lower order terms, and $f(u)$ depends only on u, not on x or t explicitly. A problem which does not satisfy these constraints is that of single phase flow in a variable area duct. If the area is $a(x)$, the equations are

$$\frac{\partial}{\partial t}(a\rho) + \frac{\partial}{\partial x}(a\rho v) = 0, \tag{1}$$

$$v_t + vv_x + \frac{1}{\rho} p_x = 0, \tag{2}$$

$$\frac{\partial}{\partial t}(a(\rho e + \tfrac{1}{2}v^2)) + \frac{\partial}{\partial x}(av(\rho e + \tfrac{1}{2}\rho v^2)) + \frac{\partial}{\partial x}(pva) = 0. \tag{3}$$

The quantities ρ, v, and e are respectively the density, velocity and internal energy averaged across the duct. In this one-dimensional approximation to three-dimensional flow, momentum is not conserved. Note that if the second equation is multiplied by $a\rho$ we have

$$\frac{\partial}{\partial t}(a\rho v) + \frac{\partial}{\partial x}(a\rho v^2) + ap_x = 0, \tag{4}$$

which is still not in conservation form.

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Chisnell [1] studied the problem of a shock propagating along a variable duct, with the gas ahead of the shock at rest. Neither the shock speed nor the state behind the shock are constant, and an exact solution is not available. Chisnell was able to derive an ordinary differential equation, $d\sigma/dx = F(\sigma, x)$, for the shock speed σ. The solution of this equation is, of course, only an *approximation* to the true shock speed. The solution is accurate if the flow behind the shock is nearly steady, and if waves generated behind the shock do not interact strongly with it.

In the next section we give a general, formal derivation of two shock propagation equations, one of which is a generalization of Chisnell's equation. In [9] another derivation is given which can also be generalized. In Section 3 we give a third, very heuristic derivation, and apply the equations to two examples one of which is the problem studied by Chisnell. In Section 4 we discuss a particular model of two-phase flow and give some of the details of the shock propagation equations for that model. The remaining details require the complete solution of the shock tube problem (Riemann problem) in a straight duct without phase changes, which is done in Section 5.

2. The shock propagation equations

We begin by considering a system of the form

$$u_t + Au_x = g(u, x)$$

with initial conditions

$$u = \begin{cases} u_l = \text{constant}, & x < 0 \\ u_r = \text{constant}, & x > 0 \end{cases}$$

and we assume $g(u_r, x) = 0$, $x > 0$, although this is not essential.

We are going to use a splitting technique which has proved useful in numerical analysis. Suppose one has an ordinary differential equation

$$\frac{dy}{dt} = b(y) + c(y) = d(y).$$

Let D be the solution operator of this equation; that is, the solution at time t with data y_0 at time 0 is

$$y(t) = D(t, y_0).$$

Let B and C be the solution operator for $dy/dt = b(y)$ and $dy/dt = c(y)$ respectively. Then, formally,

$$B(\Delta t, C(\Delta t, y_0)) = D(\Delta t, y_0) + O(\Delta t)^2.$$

This follows easily from Taylor's theorem.

Suppose now that at time $t_{n-1} = (n-1)\Delta t$ the shock is at x_{n-1} and has speed σ_{n-1}. Let the state just to the left of the shock be u_{n-1}. The first split step is to let $\bar{u}$ be the solution of

$$u_t = g(u, x_{n-1}), \quad u(0) = u_{n-1} \tag{5}$$

evaluated at $t = \Delta t$. The next step is to solve the shock tube problem with $g = 0$, left state $\bar{u}$, right state u_r, assuming that this makes sense. Assuming that a leading shock is produced, put u_n equal to the state of the left of the shock and σ_n equal to the new shock speed. See Fig. 1.

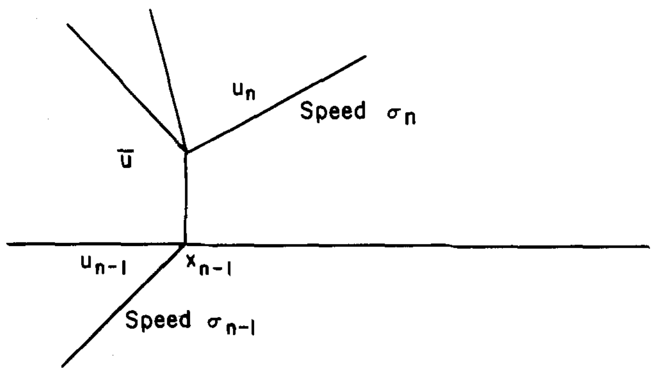

Figure 1.

To express this analytically, parametrize the shock by its speed, so that we can write the Hugoniot curve as $u = H(\sigma)$; in particular $u_{n-1} = H(\sigma_{n-1})$.

Let $S(t, u, x)$ be the solution operator of eq. (5) so that

$$\bar{u} = S(\Delta t, u_{n-1}, x_{n-1}).$$

Let $R(u)$ be the speed of the leading shock resulting from the Riemann problem with left state u, right state u_r. Then

$$\begin{aligned}\sigma_n &= R(\bar{u}) = R \cdot S(\Delta t, u_{n-1}, x_{n-1}) \\ &= R \cdot S(0, u_{n-1}, x_{n-1}) + \Delta t \frac{\partial}{\partial t} R \cdot S(0, u_{n-1}, x_{n-1}) + O(\Delta t)^2.\end{aligned}$$

But $S(0, u_{n-1}, x_{n-1}) = u_{n-1}$ and $R(u_{n-1}) = \sigma_{n-1}$, so

$$\frac{\sigma_n - \sigma_{n-1}}{\Delta t} = \frac{\partial}{\partial t} R \cdot S(0, u_{n-1}, x_{n-1}) + O(\Delta t).$$

Letting R' be the gradient vector we have

$$\frac{\partial}{\partial t} R \cdot S = R' \cdot S \cdot S_t = R' \cdot S \cdot g(u_{n-1}, x_{n-1}).$$

Formally passing to the limit,

$$\frac{d\sigma}{dt} = R'(H(\sigma)) \cdot g(H(\sigma), y),$$

where

$$y = \int_0^t \sigma(s)\,ds.$$

In terms of y, we obtain the *first shock propagation equation*,

$$\frac{d\sigma}{dy} = \frac{1}{\sigma} R'(H(\sigma)) \cdot g(H(\sigma), y). \tag{6}$$

The expression $R'(H(\sigma))$ means find $R'(u)$ for arbitrary u and then substitute $u = H(\sigma)$.

Equation (6) is the shock propagation equation associated with the shock tube. A different propagation equation can be derived for the situation depicted in Fig. 2.

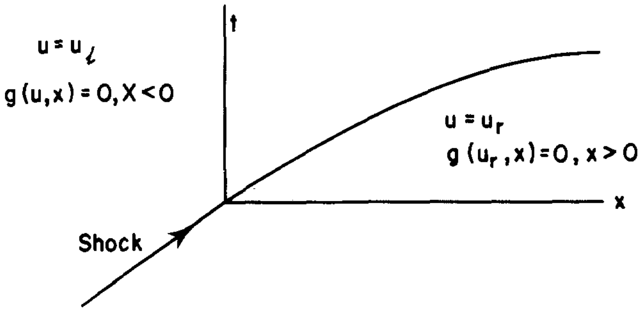

Figure 2.

In this case a shock moving at a constant speed in a region where $g = 0$ enters a region where $g \neq 0$. An example of this would be a shock moving in a tube with variable cross-section. It is easy to see that the splitting technique can be applied here by interchanging the roles of x and t. The two split equations are

$$Au_x = g \quad \text{and} \quad Au_x + u_t = 0,$$

and the resulting *second propagation equation* is

$$\frac{d\sigma}{dx} = R'(H(\sigma)) \cdot A^{-1} g(H(\sigma), x). \tag{7}$$

This is essentially the method used by Chisnell [1], as reported in Whitham [9], to describe the propagation of a hydrodynamic shock in a tube of variable cross-section. We show in the next section that eq. (7) is Chisnell's equation in that case.

3. Another derivation and some examples

We give a simpler derivation of eqs. (6) and (7) which shows more clearly the approximations that are implied. This approach was suggested to us by J. Kevorkian. Suppose there is just one differential equation. Then

$$\frac{d\sigma}{dx} = \frac{d\sigma}{du}\frac{du}{dx} = \frac{d\sigma}{du}\left(\frac{\partial u}{\partial t}\frac{1}{\sigma} + \frac{\partial u}{\partial x}\right).$$

If the state behind the shock is independent of t, then

$$\frac{d\sigma}{dx} = \frac{d\sigma}{du}\frac{\partial u}{\partial x} = \frac{d\sigma}{du}A^{-1}g.$$

On the other hand, if u is independent of x, then

$$\frac{d\sigma}{dt} = \frac{1}{\sigma}\frac{d\sigma}{du}\frac{\partial u}{\partial t} = \frac{1}{\sigma}\frac{d\sigma}{du}g.$$

In the scalar case, $d\sigma/du = R'$. In the vector case $d\sigma/du$ is not well defined, but we may define it to be R', and accept this as an approximation.

We can see now that both eqs. (6) and (7) require that waves generated behind the shock not interact too strongly with the shock, while eq. (7) requires the state behind the shock to be nearly independent of t and eq. (6) requires near independence of x.

For example, consider

$$u_t + uu_x = -u, \quad u(x, 0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}.$$

The jump condition is $\sigma(u - u_r) = (u^2 - u_r^2)$, or

$$H(\sigma) = 2\sigma - u_r, \quad R(u) = \tfrac{1}{2}(u + u_r).$$

Thus, from eq. (6),

$$\frac{d\sigma}{dt} = -\sigma, \quad \sigma(0) = \tfrac{1}{2},$$

or

$$\sigma(t) = \tfrac{1}{2}e^{-t}.$$

Indeed, the *exact* solution in this case is a shock propagating with speed $\frac{1}{2}e^{-t}$, with

$$u(x, t) = \begin{cases} e^{-t} & \text{to the left of the shock} \\ 0 & \text{to the right.} \end{cases}$$

We take up next the problem of a shock in a tube of variable cross-section $a(x)$. The equations can be written

$$\rho_t + v\rho_x + \rho u_x = -\frac{a'(x)}{a(x)}\rho v,$$

$$v_t + vv_x + \frac{1}{\rho} p_x = 0,$$

$$p_t + vp_x - c^2(\rho_t + v\rho_x) = 0,$$

where $c^2 = \partial p/\partial \rho$. Putting

$$u = \begin{pmatrix} \rho \\ v \\ p \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c^2 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ -c^2 v & 0 & v \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} -\rho v \frac{a'}{a} \\ 0 \\ 0 \end{pmatrix},$$

the system is

$$Bu_t + Cu_x = \bar{g}.$$

Therefore in eq. (7) we put $A = B^{-1}C$, $g = B^{-1}\bar{g}$. Thus, we need to find z, where $A^{-1}g = z$, or $\bar{g} = Cz$, that is,

$$-\rho v \frac{a'}{a} = vz_1 + \rho z_2, \quad 0 = vz_2 + \frac{1}{\rho} z_3, \quad 0 = -c^2 z_1 + z_3.$$

Then eq. (7) becomes

$$\frac{d\sigma}{dx} = \frac{va'}{(c^2 - v^2)a} \left[\rho v \frac{\partial R}{\partial \rho} - c^2 \frac{\partial R}{\partial v} + c^2 \rho v \frac{\partial R}{\partial p} \right]. \tag{8}$$

In eq. (8) the derivatives of R are to be evaluated as follows. Let (ρ_l, v_l, p_l) be some left state and solve the Riemann problem. The state just behind the shock is (ρ, v, p). Then compute $\partial\sigma/\partial\rho_l$, etc., and then replace ρ_l by ρ, etc. While these derivatives could be computed directly, the following observation simplifies the job. First, σ, e, u can be expressed as functions of p. Second, the structure of the rarefaction wave is such that p depends on only two parameters, the Riemann invariants $\tau = u_l + 2(\gamma p_l/\rho_l)^{\frac{1}{2}}(\gamma - 1)$ and S (S = entropy, $e\rho = p/(\gamma - 1)$)

$$\frac{\partial p}{\partial S_l} = 0 \text{ at } p = p_l, \text{ etc.}$$

Therefore

$$\frac{d\sigma}{dx} = \frac{d\sigma}{d\tau}\frac{d\tau}{dx}, \quad \frac{\partial R}{\partial v} = \frac{d\sigma}{d\tau}\frac{\partial \tau}{\partial v}, \text{ etc.,}$$

so eq. (8) becomes

$$\frac{d\tau}{dx} = -\frac{va'c}{(c+v)a}.$$

This is not Chisnell's equation; however, since the approximation is only good if the state behind the shock is nearly independent of t, $\partial S/\partial t \cong -u(\partial S/\partial x) \equiv 0$. In that case,

$$\rho c \frac{dv}{dx} + \frac{dp}{dx} = -\frac{vc^2 \rho a'}{(c+v)a},$$

with $(\rho, v, p) = H(\sigma)$. This is just eq. (8.23) of [9], which is equivalent to Chisnell's equation.

4. The variable area duct with phase changes

Many models of two-phase flow have been proposed, each having its own range of validity and each having advantages and disadvantages. A reasonably complete description of such models is given in [4]. The model used here is a simplistic one from the physical point of view; the two phases (say, gas and liquid) are assumed to be always moving at the same velocity and to have the same temperature. We call this mechanical equilibrium and thermal equilibrium. Another assumption that is made is that there is a well-defined void fraction $\alpha(x, t)$; this has the property that the integral of α over any spatial volume is the fraction of that volume occupied by gas.

The differential equations of the mechanical equilibrium-thermal equilibrium model are

$$\frac{\partial}{\partial t}(a\rho\alpha) + \frac{\partial}{\partial x}(a\rho\alpha v) = a\Phi, \tag{9}$$

$$\frac{\partial}{\partial t}(a\rho'\alpha') + \frac{\partial}{\partial x}(a\rho'\alpha' v) = -a\Phi, \tag{10}$$

$$\bar{\rho}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x}, \tag{11}$$

$$\frac{\partial}{\partial t} a(\tfrac{1}{2}\bar{\rho}v^2 + \overline{\rho e}) + \frac{\partial}{\partial x}[av(\tfrac{1}{2}\bar{\rho}v^2 + \overline{\rho e}) + apv] = 0. \tag{12}$$

The symbols used in these equations have the following meaning: α, ρ, v, e, S, T are respectively the void fraction and density, velocity internal energy, entropy, and temperature of the gas, ρ', e', S' are the corresponding quantities for the liquid, $\alpha' = 1 - \alpha$, $\bar{\rho} = \alpha\rho + \alpha'\rho'$, $\overline{\rho e} = \alpha\rho e + \alpha'\rho' e'$, $a(x)$ is the area of the duct, Φ is the rate of gas mass production per unit volume, and is assumed given as a function of the other dependent variables.

The pressure p is taken to be the gas pressure, assumed to be given by

$$p = (\gamma - 1)\exp\left[\frac{S - S_0}{c_v}\right]\rho^\gamma, \qquad c_v = \text{constant},\ S_0 = \text{constant}.$$

Then

$$(\gamma - 1)\rho e = p \text{ and } e = c_v T + e_0, \qquad e_0 = \text{constant}.$$

For the liquid we use

$$\rho' = \text{constant}, \quad e' = e'_0 + c'_v T,$$

where $e'_0 = \text{constant}$, $c'_v = \text{constant}$.

In order to write the differential equations in the form $u_t + Au_x = g$, it is convenient to use ρ, α, v, p as dependent variables. Eq. (12) becomes

$$p_t + vp_x - c^2 \frac{\Gamma}{\gamma}(\rho_t + v\rho_x) = \Lambda, \tag{13}$$

where

$$c^2 = \frac{\partial}{\partial \rho} p(\rho, s) \equiv (p_\rho)_s,$$

$$\Gamma = \frac{\gamma \rho \alpha c_v + \rho' \alpha' c'_v}{\rho \alpha c_v + \rho' \alpha' c'_v}, \tag{14}$$

$$\Lambda = (\gamma - 1)\rho c_v \cdot \frac{e' - e + p\left(\dfrac{1}{\rho'} - \dfrac{1}{\rho}\right)}{\rho \alpha c_v + \rho' \alpha' c'_v}. \tag{15}$$

Then with $u = (\rho, \alpha, v, p)^T$ and $g = (g_1, g_2, g_3, g_4)^T$ we have

$$Bu_t + Cu_x = \bar{g},$$

where

$$B = \begin{pmatrix} \alpha & \rho & 0 & 0 \\ 0 & -\rho' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -c^2 \dfrac{\Gamma}{\gamma} & 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} v\alpha & v\rho & \rho\alpha & 0 \\ 0 & -v\rho' & \rho'\alpha' & 0 \\ 0 & 0 & v & \dfrac{1}{\bar{\rho}} \\ -vc^2 \dfrac{\Gamma}{\gamma} & 0 & 0 & v \end{pmatrix},$$

$$\bar{g}_1 = -\rho\alpha v \frac{a'}{a} + \Phi, \quad \bar{g}_2 = -\rho'\alpha' v \frac{a'}{a} - \Phi, \quad \left(a' = \frac{da}{dx}\right),$$

$$\bar{g}_3 = 0, \quad \bar{g}_4 = \Lambda.$$

Thus $A = B^{-1}C$, $g = B^{-1}\bar{g}$. To compute the right side of (7) we need

$$z = A^{-1}g = C^{-1}B \cdot B^{-1}\bar{g} = C^{-1}\bar{g}.$$

The solution of this system is

$$z_3 = \frac{\Phi\left(\dfrac{1}{\rho} - \dfrac{1}{\rho'}\right) + \dfrac{\alpha\gamma}{c^2\rho\Gamma}\Lambda - v\dfrac{a'}{a}}{1 - \dfrac{v^2\alpha\bar{\rho}\gamma}{c^2\rho\Gamma}}, \quad z_4 = -\bar{\rho} v z_3.$$

It will be shown below that z_1 and z_2 are not needed.

The quantity z_3 can be simplified somewhat. Note first that (see Section 5) the speed of sound in this model is

$$c_T^2 = \frac{c^2 \rho \Gamma}{\alpha \bar{p} \gamma} = \frac{\Gamma p}{\alpha \bar{p}},$$

so that the denominator of z_3 is $1 - v^2/c_T^2$.

Using the definition of Λ and rearranging terms, we may write

$$z_3 = \left(1 - \frac{v^2}{c_T^2}\right)^{-1} \left[(X + Y)\Phi - v\frac{a'}{a}\right], \tag{16}$$

where

$$X = \frac{1}{\Gamma}\left(\frac{1}{\rho} - \frac{1}{\rho'}\right), \tag{17}$$

$$Y = \frac{\alpha c_v(\gamma - 1)\rho}{p(\gamma \rho \alpha c_v + \rho' \alpha' c_v')}(e' - e). \tag{18}$$

In order to obtain the first propagation equation (6), we need $g = B^{-1}\bar{g}$. In fact, all we will need are g_3 and g_4, which are

$$g_3 = 0, \quad g_4 = \Lambda + \frac{c^2 \Gamma \rho}{\gamma \alpha}\left[\Phi\left(\frac{1}{\rho} - \frac{1}{\rho'}\right) - v\frac{a'}{a}\right] = c_T^2 \bar{\rho}\left[(X + Y)\Phi - v\frac{a'}{a}\right].$$

5. The complete shock propagation equations

The first shock propagation equation (6) becomes

$$\sigma \frac{d\sigma}{dx} = \frac{\partial R}{\partial \alpha} g_1 + \frac{\partial R}{\partial \alpha} g_2 + \frac{\partial R}{\partial v} g_3 + \frac{\partial R}{\partial p} g_4 = R' \cdot g,$$

while eq. (7) becomes

$$\frac{d\sigma}{dx} = R' \cdot z.$$

We remind the reader what R' is: the Riemann problem for a fixed right state and arbitrary left state ρ_l, α_l, v_l, p_l is resolved and supposed to produce a leading shock with speed $\sigma = R = R(\rho_l, \alpha_l, v_l, p_l)$. Frpm this we calculate $\partial R/\partial \rho_l$, etc. However, this must be evaluated *at the shock;* that is, if (ρ, α, v, p) is the state just behind the shock, then we need $R'(\rho, \alpha, v, p)$. To obtain this gradient we now refer to the equations and notation of the shock tube solution given in Section 6. We note that by eq. (41) the shock speed depends only on the pressure p behind the shock, which is determined by

$$\tau = \psi(p) + (\tau - v_l)\left(\frac{p}{p_l}\right)^{(\Gamma_l - 1)/2\Gamma_l}$$

where

$$\tau = v_l + \frac{2}{\Gamma_l - 1}\left(\frac{\alpha \Gamma p}{\bar{\rho}}\right)_l^{\frac{1}{2}}$$

Therefore, at $p = p_l$, $v = v_l$, $\rho = \rho_l$, $\alpha = \alpha_l$, we see immediately that

$$\frac{\partial p}{\partial \rho} = \frac{\partial p}{\partial \alpha} = 0.$$

Further calculation shows that

$$\frac{\partial p}{\partial v} = h$$

and

$$\frac{\partial p}{\partial p_l} = h\left(\frac{\alpha}{\Gamma p \bar{\rho}}\right)^{\frac{1}{2}} = \frac{h}{c_T \bar{\rho}}, \text{ at } p = p_l,$$

where

$$h = \left(\psi'(p) + \frac{1}{c_T \bar{\rho}}\right)^{-1}.$$

We now readily obtain the propagation equations.

First shock propagation equation:

$$\frac{d\sigma}{dx} = \frac{d\sigma}{dp} h c_T \sigma^{-1}\left[(X + Y)\Phi - v\frac{a'}{a}\right]. \tag{19}$$

Second (Chisnell) shock propagation equation:

$$\frac{d\sigma}{dx} = \frac{d\sigma}{dp} h\left(1 + \frac{v}{c_T}\right)^{-1}\left[(X + Y)\Phi - v\frac{a'}{a}\right]. \tag{20}$$

The right sides of these equations must be expressed as functions of σ, that is, p, l, α, and v must be written as functions of σ. These functions are obtained from the shock jump conditions. We get

$$\frac{d\sigma}{dp} = [2\sigma(1 - \bar{\mu}^2)\alpha_r \bar{\rho}_r]^{-1}, \tag{21}$$

$$p = \sigma^2\alpha_r\bar{\rho}_r(1 - \bar{\mu}^2) - \bar{\mu}^2 p_r, \tag{22}$$

$$v = \frac{p - p_r}{\sigma\bar{\rho}_r}, \tag{23}$$

$$\rho = \rho_r \frac{p + \bar{\mu}^2 p_r}{p_r + \bar{\mu}^2 p}, \tag{24}$$

$$\alpha = \frac{\rho_r\alpha_r}{\rho\alpha_r' + \rho_r\alpha_r}. \tag{25}$$

A short calculation shows that

$$h = \left(\frac{p + p_r(1 + 2\bar{\mu}^2)}{2\bar{\rho}_r\sigma(p + \bar{\mu}^2 p_r)} + \frac{1}{c_T\bar{\rho}}\right)^{-1}. \tag{26}$$

The quantities X and Y are given by equations (16) and (17).

We close this section with the following observation. If

$$\alpha c_v e' + e\alpha' c_v' \rho'/\rho > e\alpha c_v \rho/\rho' + e\alpha' c_v',$$

(e.g., if $\rho' > \rho$ and $\rho' e' > \rho e$) then $X + Y > 0$. Since $d\sigma/dp$ and h are positive, this means that the coefficient of Φ is positive (if $v/c > -1$). Thus, condensation behind the shock will slow it down, evaporation will speed it up.

6. The shock tube

We present in this section the complete solution of the shock tube problem, without proofs. First, for completeness, we restate the differential equations and equations of state

$$\frac{\partial}{\partial t}(\alpha\rho) + \frac{\partial}{\partial x}(\alpha\rho v) = 0,$$

$$\frac{\partial}{\partial t}(\alpha'\rho') + \frac{\partial}{\partial x}(\alpha'\rho' v) = 0,$$

$$\frac{\partial}{\partial t}(\bar{\rho} v) + \frac{\partial}{\partial x}(\bar{\rho} v^2) + \frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\bar{\rho} v^2 + \overline{\rho e}) + \frac{\partial}{\partial x} v(\tfrac{1}{2}\bar{\rho} v^2 + \overline{\rho e}) + \frac{\partial}{\partial x} pv = 0.$$

Unprimed symbols refer to the gas, primed symbols refer to the liquid. Then ρ, v, e are respectively density, velocity, and internal energy. The void fraction is α, and $\alpha' = 1 - \alpha$. The quantity $\bar{\rho}$ is the mixture density,

$$\bar{\rho} = \alpha\rho + \alpha'\rho'.$$

Also

$$\overline{\rho e} = \rho\alpha e + \rho'\alpha' e'.$$

The equation of state of the gas is

$$p = (\gamma - 1)\exp\left[\frac{s - s_0}{c_v}\right]\rho^{\gamma}$$

where

$$c_v = \text{constant},\ s_0 = \text{constant},\ s = \text{entropy}.$$

The equation of state of the liquid is assumed to be

$$e' = e_0'' + \frac{c_v'}{c_v} e$$

where, $e_0'' = \text{constant}$, $c_v' = \text{constant}$.

The differential equations are a hyperbolic system of conservation laws. There are four characteristic speeds:

$$\lambda_1 = v - c_T,\ \lambda_2 = \lambda_3 = v,\ \lambda_4 = v + c_T,$$

where

$$c_T = \left(\frac{\Gamma p}{\alpha \bar{\rho}}\right)^{\frac{1}{2}}, \quad \Gamma = \frac{\gamma p \alpha c_v + \rho'\alpha' c_v'}{\rho\alpha c_v + \rho'\alpha' c_v'}.$$

If the left state has a higher density and lower void fraction than the right state, the solution of the shock tube problem is as shown in Fig. 3.

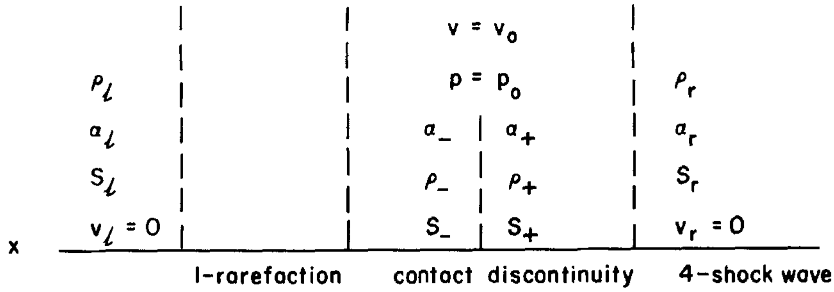

Figure 3. Solution at $t > 0$.

The 1-rarefaction wave corresponds to the eigenvalue $v - c_T$, the 4-shock wave corresponds to the eigenvalue $v + c_T$.

In general, the solution is obtained by equating the velocity and pressure obtained from the rarefaction wave to the velocity and pressure obtained from the shock wave. This determines ρ_-, ρ_+, and then the remaining variables can be computed.

To find the intermediate states, let

$$\bar{\mu}^2 = \frac{\Gamma_r - 1}{\Gamma_r + 1}, \tag{27}$$

$$\phi(p) = -\left(\frac{2\alpha c_T}{\Gamma - 1}\right)_l \left[\left(\frac{p}{p_l}\right)^{(\Gamma_l - 1)/2\Gamma_l} - 1\right], \tag{28}$$

$$\psi(p) = ((1 - \bar{\mu}^2)\alpha_r)^{\frac{1}{2}} (\bar{\rho}_r(p + \bar{\mu}^2 p_r))^{-\frac{1}{2}} (p - p_r) \tag{29}$$

Then, for p_0,

$$v_l + \phi(p_0) = \psi(p_0), \tag{30}$$

$$v_0 = \psi(p_0), \tag{31}$$

$$\rho_+ = \rho_r \frac{p_0 + \bar{\mu}^2 p_r}{p_r + \bar{\mu}^2 p_0}, \tag{32}$$

$$\alpha_+ = \frac{\rho_r \alpha_r}{\rho_+(1 - \alpha_r) + \rho_r \alpha_r}, \tag{33}$$

$$\rho_- = \left(\frac{p_0}{p_l}\right)^{1/\Gamma_l} \rho_l, \tag{34}$$

$$\alpha_- = \frac{\rho_l \alpha_l}{\rho_-(1 - \alpha_l) + \rho_l \alpha_l}. \tag{35}$$

To find the solution (ρ, α, v, p) at a point (x, t) inside the rarefaction wave, let $x/t = \eta$. Then the following equations determine (ρ, α, v, p) at (x, t):

$$v - c_T = \eta, \tag{36}$$

$$v_l + \phi(p) = v, \tag{37}$$

$$\frac{\rho_l \alpha_l}{\rho' \alpha_l'} = \frac{\rho \alpha}{\rho' \alpha'}, \tag{38}$$

$$p = p_l \left(\frac{\rho}{\rho_l}\right)^\Gamma, \tag{39}$$

$$\Gamma = \Gamma_l. \tag{40}$$

Finally, the shock speed is

$$\sigma^2 = \frac{p_0 + \bar{\mu}^2 p_r}{(1 - \bar{\mu}^2)\alpha_r \bar{\rho}_r}. \tag{41}$$

Many papers have been written on acoustic, rarefaction, and shock waves in two-phase flow. A fairly complete survey up to 1968 is given in [3], while van Wijngaarden [6] gives more recent references. A derivation of the equations in this section is contained in [8].

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